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Subject: Digital Signal Processing EC 601

Analog Chebyshev Low pass filter:

Two Types of filters:

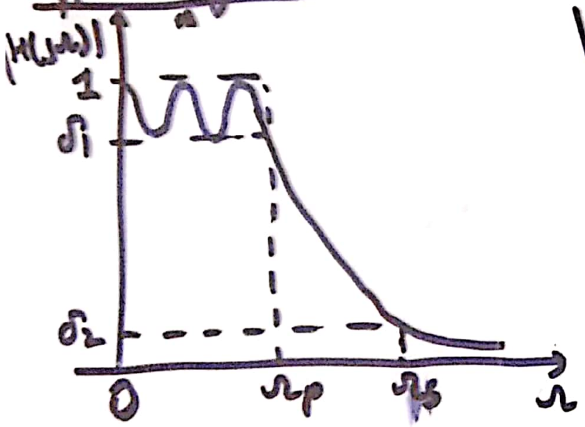
(1) Type I Chebyshev filter :→ All pole filters

→ Exhibits equiripple behaviour in passband and monotonic characteristic in stopband.

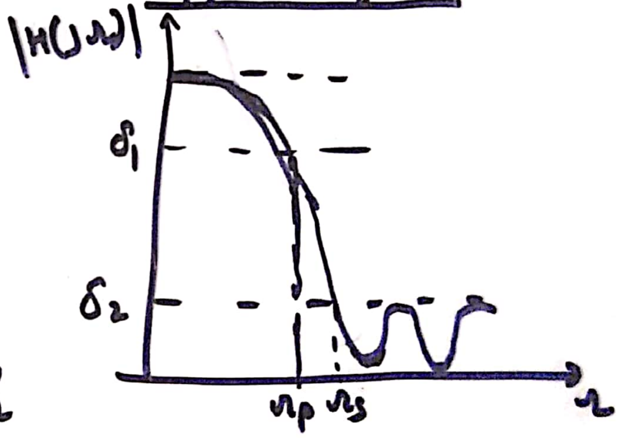
(2) Type II Chebyshev filter: → Contain both poles and zeros

→ exhibits monotonic behaviour in passband and an equiripple behaviour in stopband.

Type I filter.



Type II filter



TYPE I CHEBYSHEV FILTER:

The magnitude squared response of Nth order type I Chebyshev filter is

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\Omega/\Omega_c)}$$

ε → passband Ripple parameter

$C_N(x)$ is Nth order Chebyshev Polynomial.

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$$C_N(x) = C_N\left(\frac{\Omega}{\Omega_c}\right) = \begin{cases} \cos(N\cos^{-1}x) , & |x| \le 1 \text{ (Passband)} \\ \cosh(N\cosh^{-1}x) , & |x| > 1 \text{ (Stopband)} \end{cases}$$

We can obtain higher order Chebyshev Polynomial from recursive relation

$$C_N(x) = 2x\,C_{N-1}(x) - C_{N-2}(x) , \quad N > 1$$

$$C_0(x) = 1 \quad \text{and} \quad C_1(x) = x$$

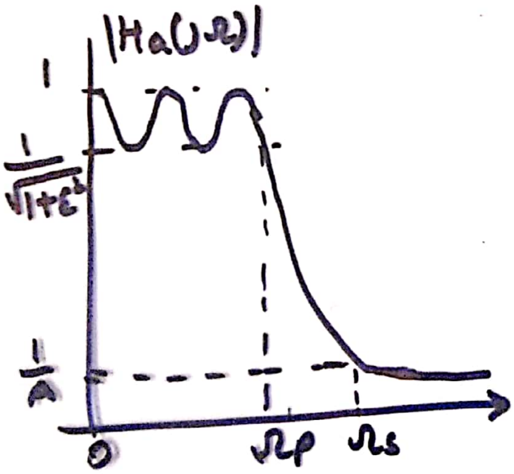
$x = \frac{\Omega}{\Omega_c}$ is normalised frequency.

→ For $|x| \le 1$, $C_N(x)$ oscillates with equal ripple &

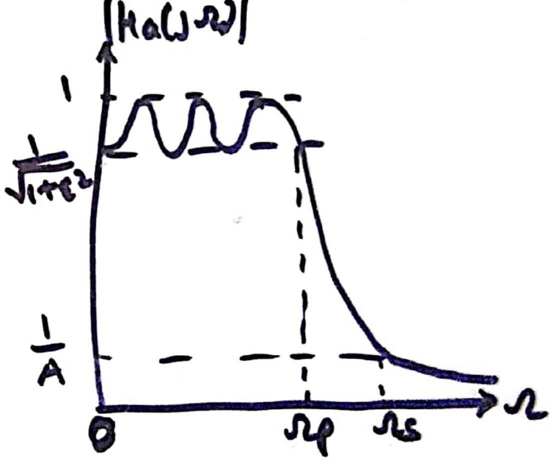
→ For $|x| > 1$, $C_N(x)$ increases monotonically to ∞.

→ $C_N(1) = 1$ for all N.

→ There are two possible shapes of $|H_a(j\Omega)|^2$,



N odd



N even

$|H_a(j0)| = 1$ for N odd

$C_N(0) = 0$ for N odd

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$|Ha(j0)| = \frac{1}{\sqrt{1+\epsilon^2}}$ for N even. $\quad x = \frac{\Omega}{\Omega_c}$

$\therefore C_N(0) = 1$ for N even

→ For $0 \le x \le 1$, (or $0 \le \Omega \le \Omega_c$), $|Ha(j\Omega)|$ oscillates between 1 and $\frac{1}{\sqrt{1+\epsilon^2}}$

→ For $x > 1$ ($\Omega > \Omega_c$), $|Ha(j\Omega)|$ decreases monotonically to 0.

→ At $x = \frac{\Omega_s}{\Omega_c}$ (or $\Omega = \Omega_s$), $|Ha(j\Omega_s)| = \frac{1}{A}$

To determine a causal and stable Ha(s), we find the poles of Ha(s) Ha(-s) and select the plane poles in the left half for Ha(s).

$$|Ha(j\Omega)|^2\Big|_{\Omega = \frac{s}{j}} = Ha(s)Ha(-s) = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{s}{j\Omega_c}\right)}$$

$$1 + \epsilon^2 C_N^2\left(\frac{s}{j\Omega_c}\right) = 0 \Rightarrow C_N^2\left(\frac{s}{j\Omega_c}\right) = 0 - \frac{1}{\epsilon^2}$$

$$C_N\left(\frac{s}{j\Omega_c}\right) = 0 \pm j\frac{1}{\epsilon}$$

For $\Omega < \Omega_c$ $\quad \cos(N\cos^{-1}x)$

$$C_N\left(\frac{s}{j\Omega_c}\right) = \cos\left[N\cos^{-1}\left(\frac{s}{j\Omega_c}\right)\right] = \cos N\theta$$

$\theta \rightarrow$ complex number

$$\cos^{-1}\left(\frac{s}{j\Omega_c}\right) = \theta = u + jv \quad \text{— (A)}$$

$$\cos N\theta = \cos(Nu + jNv) = \cos Nu \cos jNv + \sin Nu \sin jNv$$

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$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin j\theta = \frac{e^{j(j\theta)}-e^{-j(j\theta)}}{2j}$

$= \frac{e^{-\theta}-e^{\theta}}{2j} = -j\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)$

$= -j\sinh\theta$

$\cos j\theta = \frac{e^{j(j\theta)}+e^{-j(j\theta)}}{2}$

$= \frac{e^{-\theta}+e^{\theta}}{2} = \cosh\theta$

$\cos N\theta = \cos Nu \cosh Nv - j\sin Nu \sinh Nv = 0 \pm j\frac{1}{\epsilon}$

Equating the real & imaginary parts

$\cos Nu \cosh Nv = 0$

$\sin Nu \cdot \sinh Nv = \pm\frac{1}{\epsilon}$

$\cosh(Nv) > 0$ its smallest value is 1

$\therefore \cos Nu = 0$

$u = \frac{\pi}{2N}, \frac{3\pi}{2N}, \frac{5\pi}{2N}, ---$

$u = \frac{\pi}{2N}(2K+1)$, $K = 0, 1 --- 2N-1$

For these values of u, $\sin Nu = \pm 1$

$\sinh Nv = \pm\frac{1}{\epsilon}$

$v = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right) = \pm a$ (say)

From (A), $s = j\Omega_c \cos\theta$

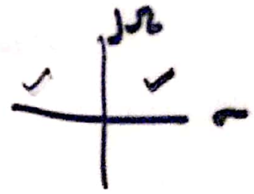
$P_K = j\Omega_c \cos\theta = \sigma + j\Omega = j\Omega_c \cos\left(\frac{(2K+1)}{2N}\pi \pm ja\right)$

$\theta = u + jv$

$\sigma + j\Omega = j\Omega_c \cosh a \cos\frac{\pi}{2N}(2K+1) \pm \Omega_c \sinh a \sin\frac{\pi}{2N}(2K+1)$

$\sigma = \pm\Omega_c \sinh a \sin\frac{\pi}{2N}(2K+1)$

$\Omega = \Omega_c \cosh a \cos\frac{\pi}{2N}(2K+1)$



Taking LHS poles

$\sigma = -\Omega_c \sinh a \sin\frac{\pi}{2N}(2K+1)$

$\Omega = \Omega_c \cosh a \cos\frac{\pi}{2N}(2K+1)$ $K = 0, 1 -- N-1$

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$$-\frac{\sigma}{\Omega_c \sinh a} = \frac{\sin \pi(2k+1)}{2N} \quad -(1)$$

$$\frac{\Omega}{\Omega_c \cosh a} = \frac{\cos \pi(2k+1)}{2N} \quad -(2)$$

squaring and adding (1) & (2),

$$\left(\frac{\sigma}{\Omega_c \sinh a}\right)^2 + \left(\frac{\Omega}{\Omega_c \cosh a}\right)^2 = 1$$

this is the equation of an ellipse with major axis $\Omega_c \cosh a$ and minor axis $\Omega_c \sinh a$.

The system function is

$$H_a(s) = \frac{K}{\prod (s - p_k)} \quad (s-p_0)(s-p_1)\cdots$$

K is a normalising factor choosen to make

$$H_a(j0) = \begin{cases} 1 & \text{for odd } N \\ \dfrac{1}{\sqrt{1+\epsilon^2}} & \text{for even } N \end{cases}$$

Determination of Filter Parameters (order N and cut off frequency Ω_c) of Chebyshev filter:

Case I: When the specifications are ε, A, Ω_p and Ω_s

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

Desired specifications of LPF are

$$\delta_1 = \frac{1}{1+\varepsilon^2} \le |H_a(j\Omega)|^2 \le 1 \qquad 0 \le \Omega \le \Omega_p$$

$$|H_a(j\Omega)| \le \delta_2 \qquad \Omega_s \le \Omega \le \pi$$

At $\Omega = \Omega_p$, $|H_a(j\Omega_p)|^2 = \dfrac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega_p}{\Omega_c}\right)} = \delta_1^2$

Assume $\Omega_c = \Omega_p$

$$\delta_1^2 = \frac{1}{1+\varepsilon^2}, \quad C_N\left(\frac{\Omega_c}{\Omega_c}\right) = C_N(1) = 1$$

$$\varepsilon = \left(\frac{1}{\delta_1^2} - 1\right)^{0.5}$$

At $\Omega = \Omega_s$, $|H_a(j\Omega_s)|^2 = \dfrac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \delta_2^2$

Since $\Omega_s > \Omega_p$

$$C_N\left(\frac{\Omega_s}{\Omega_p}\right) = \cosh\left(N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)$$

$$\delta_2^2 = \frac{1}{1+\varepsilon^2\left(\cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)\right)^2}$$

$$\cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right) = \frac{1}{\varepsilon}\left(\frac{1}{\delta_2^2}-1\right)^{0.5}$$

$$N = \frac{\cosh^{-1}\left[\frac{1}{\varepsilon}\left(\frac{1}{\delta_2^2}-1\right)^{0.5}\right]}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad \Omega_c = \Omega_p$$

<u>Case II</u> : When the specifications are A_p, A_s, Ω_p and Ω_s.

$A_p \rightarrow$ Max. Passband attenuation

$A_s \rightarrow$ Min. Stopband attenuation

The magnitude specifications (in dB) are

$$-A_p \le |H_a(j\Omega)|_{dB} \le 0 \;,\; 0 \le \Omega \le \Omega_p$$

$$|H_a(j\Omega)|_{dB} \le -A_s ,\; \pi \ge |\Omega| \ge \Omega_s$$

At $\Omega = \Omega_p$, $-A_p = 20\log_{10}|H_a(j\Omega_p)|$

$$-A_p = 10\log_{10}|H_a(j\Omega_p)|^2$$

$$-A_p = 10\log_{10}\frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega_p}{\Omega_c}\right)}$$

$$A_p = -10\log_{10}\cdot\frac{1}{1+\varepsilon^2} \quad (\text{Assume } \Omega_c = \Omega_p)$$

$10^{-A_p/10}$

$-\frac{A_p}{10} = \log_{10}\frac{1}{1+\varepsilon^2}$

$$\varepsilon = \sqrt{10^{A_p/10}-1}$$

At $\Omega = \Omega_s$

$$-A_s = 20\log_{10}|H_a(j\Omega_s)| = 10\log_{10}\cdot\frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_c}\right)}$$

$$A_s = -10\log_{10}\cdot\frac{1}{1+\varepsilon^2 C_N^2(\Omega_s/\Omega_c)}$$

$$C_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{10^{A_s/10} - 1}{\varepsilon^2} = \frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}$$

$$C_N\left(\frac{\Omega_s}{\Omega_p}\right) = \sqrt{\frac{10^{0.1A_s} - 1}{10^{0.1A_p} - 1}} \qquad \Omega_s > \Omega_p$$

$$\cosh\left(N \cosh^{-1}\frac{\Omega_s}{\Omega_p}\right) = \sqrt{\frac{10^{0.1A_s} - 1}{10^{0.1A_p} - 1}}$$

$$N = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{0.1A_s} - 1}{10^{0.1A_p} - 1}}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad \Omega_c = \Omega_p$$

Design Procedure:

1) Transform the magnitude response specifications of digital filter into analog filter specifications by using either impulse invariant or Bilinear transformation method.
2) Calculate the filter order N. The cut off frequency $\Omega_c = \Omega_p$
3) Determine $a = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) = \frac{1}{N}\ln\left(\frac{1}{\varepsilon} + \sqrt{1 + \frac{1}{\varepsilon^2}}\right)$

   $\left[\because \sinh^{-1}x = \ln\left[x + \sqrt{1+x^2}\right]\right]$
4) compute $\sinh a$ and $\cosh a$
5) Find the poles.
6) Determine the analog filter transfer function $H_a(s)$
7) Transform $H_a(s)$ to $H(z)$.

Example: Design a Chebyshev digital IIR LPF using Bilinear transformation to satisfy the following specifications

Passband : $0.8 \le |H(e^{j\omega})| \le 1$, $0 \le \omega \le 0.2\pi$

stopband $|H(e^{j\omega})| \le 0.2$, $0.32\pi \le \omega \le \pi$

Sol $\delta_1 = 0.8$, $\delta_2 = 0.2$, $\omega_p = 0.2\pi$, $\omega_s = 0.32\pi$

$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 0.65$ rad/s

$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 1.1$ rad/s

$$N = \frac{\cosh^{-1}\left[\frac{1}{\varepsilon}\left(\frac{1}{\delta_2^2}-1\right)^{0.5}\right]}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \quad , \varepsilon = \left(\frac{1}{\delta_1^2}-1\right)^{0.5} = 0.75$$

$N = 2.2941 \simeq 3$, $\Omega_c = \Omega_p = 0.65$ rad/s

$$a = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) = \frac{1}{N}\ln\left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2}+1}\right)$$

$$= \frac{1}{3}\ln\left(\frac{1}{0.75} + \sqrt{\frac{1}{0.5625}+1}\right) = 0.3662$$

$\sinh a = 0.3744$, $\cosh a = 1.0678$ $K = 0, 1 \ldots N-1$

$\sigma_k = -\Omega_c \sinh a \sin\frac{\pi(2k+1)}{2N}$ $K = 0, 1, 2$

$\Omega_k = \Omega_c \cosh a \cos\frac{\pi(2k+1)}{2N}$ $s_k = \sigma_k + j\Omega_k$

$\sigma_0 = -0.65 \times 0.3744 \times \sin\frac{\pi}{6} = -0.1217$

$\Omega_0 = 0.65 \times 1.0678 \times \cos\pi/6 = 0.6011$

$$\sigma_1 = -0.65 \times 0.3744 \times \sin\frac{3\pi}{6} = -0.2434$$

$$\Omega_1 = 0.65 \times 1.0678 \times \cos\frac{3\pi}{6} = 0$$ 3

$$\sigma_2 = -0.65 \times 0.3744 \times \sin 5\pi/6 = -0.1217$$

$$\Omega_2 = 0.65 \times 1.0678 \times \cos 5\pi/6 = -0.6011$$

The poles are

$$P_0 = \sigma_0 + j\Omega_0 = -0.1217 + j\,0.6011$$

$$P_1 = \sigma_1 + j\Omega_1 = -0.2434 + j0$$

$$P_2 = \sigma_2 + j\Omega_2 = -0.1217 - j\,0.6011 = P_0^*$$

$$H_a(s) = \frac{K}{\prod_K (s-P_K)} = \frac{K}{(s-P_0)(s-P_1)(s-P_2)}$$

$$= \frac{K}{(s+0.2434)(s-P_0)(s-P_0^*)}$$

$$H_a(s) = \frac{K}{(s+0.2434)(s^2+0.2434s+0.3761)}$$

$$H_a(j0) = \frac{K}{0.2434 \times 0.3761} = 1 \quad \left[\because H_a(j0) = 1 \text{ for odd } N\right]$$

$$K = 0.2434 \times 0.3761 = 0.0915$$

$$\therefore H_a(s) = \frac{0.0915}{(s+0.2434)(s^2+0.2434s+0.3761)}$$

Using Bilinear transformation,

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{0.0084 + 0.0252z^{-1} + 0.0252z^{-2} + 0.0084z^{-3}}{1 - 2.2735z^{-1} + 1.9668z^{-2} - 0.6262z^{-3}}$$